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EXTREMUM PRINCIPLES FOR A CLASS OF DYNAMIC RIGID-PLASTIC PROBLEMS†

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Abstract—A number of extremum principles are developed for the acceleration fields associated with rigid-plastic dynamic loading problems. Major emphasis is given to global and local principles for mode solutions (where the spatial variables may be separated from time). The extremum principles are developed by first considering a simple model and then proved for structures of any configuration. The discussion is limited to problems in which the displacements are small.

1. INTRODUCTION

IN RECENT years Martin and Symonds [1] suggested a means of obtaining approximate solutions to rigid-plastic impulsive loading problems by using mode solutions, i.e. solutions in which the shape of the velocity field remains constant. Martin and Symonds showed that the actual solution converged on the mode solution and that the accelerations in the mode solution were independent of time. However, they did not provide any formal method by which the mode acceleration field could be obtained. In applying similar methods to rigid-viscoplastic problems, Lee and Martin [2] showed that mode solutions could be obtained by a variational principle. Variational principles cannot be applied with ease to rigid-plastic problems because of the presence of rigid regions and singularities in the strain rate field and the original intention of the work reported in this paper was to attempt to show by alternative methods that a result similar in form to that given by Lee and Martin could be applied to rigid-plastic problems.

In the course of this work it was discovered that a particularly simple two degree of freedom model, which can be constructed to illustrate the behavior of a dynamically loaded rigid-plastic structure, provided a means of suggesting the form of the result which was sought. This model has previously been used by Rawlings [3, 4] and Nayfeh and Prager [5]. In addition to the result which was desired the model indicated the existence of a number of other hitherto unnoticed extremum principles for the general dynamic problem and the problem of a structure deforming dynamically under constant loads. The model is explained in brief terms and the extremum principles suggested by it are proved in the general case.

2. THE RIGID-PLASTIC DYNAMIC LOADING PROBLEM

Consider a class of structures which consists either of assemblages of curved or straight bars (one-dimensional) or thin sheets of a given configuration (two-dimensional). In the

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one-dimensional case we neglect the cross section dimensions of the bars in comparison with their length and locate any point on the structure by means of one space variable measured along the center-lines of the bars. In the two-dimensional case we neglect the thickness of the sheet in comparison with the other dimensions and locate a point on the structure by means of two space variables measured on the middle surface of the sheet. In either case we can adequately represent the space variables by s and the domain occupied by the structure by S. Let the specific mass of the structure be m.

The external forces acting on the structure can be represented by $\mathbf{p}(s, t)$ (per unit length or per unit area) and we permit $\mathbf{p}(s, t)$ to include delta functions. Further we treat $\mathbf{p}(s, t)$ as generalized forces, including both direct forces and couples. $\mathbf{p}(s, t)$ can thus represent distributed forces, distributed couples, point forces and point couples. We define the conjugate generalized velocities $\dot{\mathbf{u}}(s, t)$ which include both displacement rates and rotation rates. The internal forces or generalized stresses can be represented by Q_j $(j = 1, \ldots, n)$ and the generalized strain rates by $\dot{q}_j$.

The classical small displacement assumptions will be adopted. Specifically, we assume that accelerations $\ddot{\mathbf{u}}(s, t)$ can be computed as the partial derivatives of $\dot{\mathbf{u}}$ with respect to time t, and that the dynamic equations are written in the original configuration and are linear in $\mathbf{p}(s, t)$, $Q_j(s, t)$ and $\ddot{\mathbf{u}}(s, t)$. The kinematic relations, which permit the generalized strain rates $\dot{q}_j$ to be derived from the generalized velocities $\dot{\mathbf{u}}$ and include any appropriate compatibility relations, are also assumed to be linear. These assumptions permit the principle of virtual velocities to be written. If $\mathbf{p}$, Q_j, $\ddot{\mathbf{u}}$ satisfy the dynamic equations they are said to be *dynamically admissible*. If $\dot{q}_j(s, t)$, $\dot{\mathbf{u}}(s, t)$ satisfy the kinematic relations they are said to be *kinematically admissible*. For any dynamically admissible set $\mathbf{p}(s, t)$, $Q_j(s, t)$, $\ddot{\mathbf{u}}(s, t)$ and any kinematically admissible set $\dot{q}_j^*(s, t)$, $\dot{\mathbf{u}}(s, t)$ we have

$$\int_S \mathbf{p}(s) \,.\, \dot{\mathbf{u}}^*(s)\,\mathrm{d}s - \int_S m\ddot{\mathbf{u}} \,.\, \dot{\mathbf{u}}^*\,\mathrm{d}s = \int_S Q_j(s)\dot{q}_j^*(s)\,\mathrm{d}s. \tag{1}$$

The structure is assumed to be composed of a rigid, perfectly plastic material. The constitutive relation is

$$\dot{q}_j = \lambda \frac{\partial \phi}{\partial Q_j} \tag{2}$$

where $\phi = \phi(Q_j)$ is a convex yield function

$$\begin{aligned} &\lambda \geq 0 \quad \text{for } \phi = 0 \quad \text{and } \frac{\partial \phi}{\partial Q_j}\dot{Q}_j = 0 \\ &\lambda = 0 \quad \text{for } \phi < 0 \\ &\qquad\quad \text{or } \phi = 0 \quad \text{and } \frac{\partial \phi}{\partial Q_j}\dot{Q}_j < 0 \\ &\qquad\quad \phi(Q_j) > 0 \text{ is not admitted.} \end{aligned}$$

Equation (2) applies only if $\phi(Q_j)$ is continuously differentiable; if corners are present in the yield surface the outward normal vector to the yield surface in stress space, $\partial\phi/\partial Q_j$ is not uniquely defined, but the direction of the strain rate vector $\dot{q}_j$ is constrained to lie between adjacent normals to the yield surface. We note that it follows from the convexity of the yield surface and the normality rule that for two states of stress Q_j, Q_j^*, which satisfy the restrictions $\phi(Q_j) \leq 0$, $\phi(Q_j^*) \leq 0$ and their associated strain rates $\dot{q}_j$, $\dot{q}_j^*$,

$$(Q_j^* - Q_j)\dot{q}_j^* \geq 0, \qquad (Q_j - Q_j^*)\dot{q}_j \geq 0,$$

and

$$(Q_j - Q_j^*)(\dot{q}_j - \dot{q}_j^*) \geq 0. \tag{3}$$

If the strain rate $\dot{q}_j$ is given, the stress Q_j associated with the strain rate will be uniquely determined if the yield surface is strictly convex. If the yield surface contains flat regions, Q_j may not be uniquely determined. In all cases, however, the specific rate of dissipation of energy, $D(\dot{q}_j)$, defined by

$$D(\dot{q}_j) = Q_j\dot{q}_j \tag{4}$$

is uniquely determined when $\dot{q}_j$ is given.

The dynamic loading problem is given as follows. To specify the boundary conditions, we give loads $\hat{\mathbf{p}}(s, t)$, or components of the loads, on part of S which we denote by S_p. The velocity $\dot{\mathbf{u}}(s, t)$, or components of the velocity, are set equal to zero on the remainder of S, which we denote by S_u. The initial conditions consist of initial velocities $\dot{\mathbf{u}}(s, 0)$ on S_p at time $t = 0$, with the initial displacements taken to be zero everywhere.

The solution of the problem involves the determination of the velocity field $\dot{\mathbf{u}}(s, t)$ on S_p, the reactions $\mathbf{p}(s, t)$ on S_u, the stresses $Q_j(s, t)$ and the strain rates $\dot{q}_j(s, t)$. The uniqueness of the solution has been discussed by Martin [6]; we expect that the velocity field will be unique, while the stresses may not be determined uniquely in rigid regions of the body or where flats on the yield surface occur.

The first problem we shall consider is that of determining the acceleration field $\ddot{\mathbf{u}}(s, t)$ at an instant when the velocity field $\dot{\mathbf{u}}(s, t)$ is known. It is evident that if this problem can be solved, the entire dynamic solution can be obtained by considering successive instants in which the velocity field is obtained by difference formulae from the velocities and accelerations at preceding instants.

The second problem which will be studied is the simpler problem, which will be referred to as the step load problem, in which the loads $\hat{\mathbf{p}}(s, t)$ on S_p are independent of time, i.e. $\hat{\mathbf{p}}(s, t) = \hat{\mathbf{p}}(s)$. In all other respects the problem is identical to that laid out above.

In considering these problems it is our intention to treat first a simple model which contains all the important elements which contribute to the mechanical behavior. The model will be seen to suggest new extremum principles for the dynamic rigid-plastic problem, and we shall then proceed to prove these principles without reference to the model.

We shall be concerned first with the general problem of determining the instantaneous acceleration field in a structure in which the instantaneous velocity field and external loading is known. A dynamic (as opposed to kinematic) extremum principle is derived. Thereafter we shall consider the problem of determining velocity fields which lead to *mode solutions* for constant external loads, i.e. solutions in which the velocity field may be written as a product of independent functions of space and time. Both dynamic and kinematic principles are derived and the principles may be given in terms of either the acceleration field in the mode solution or the velocity field. These principles are not strict variational principles, but are characterized by a functional whose first variation is either non-positive or non-negative.

3. A SIMPLE DYNAMIC MODEL

Consider the simple beam shown in Fig. 1. The beam is composed of a rigid-plastic material. The beam itself is assumed to be massless, but two *equal* lump masses m are attached to it. External loads, P_α ($\alpha = 1, 2$) can be applied to the masses in the vertical plane and transverse to the beam. The transverse velocities of the masses are $\dot{u}_\alpha$ ($\alpha = 1, 2$).

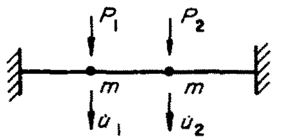

FIG. 1.

When the beam is loaded quasi-statically (with all inertia terms zero) flow will occur when the loads take on certain values. These values can be represented by a *limit surface* $\psi(P_\alpha) = 0$ in the P_α space (Fig. 2). When the load state P_α lies within the limit surface, the beam will be rigid and we arrange ψ so that $\psi(P_\alpha) < 0$ for such states. When the loads are such that $\psi(P_\alpha) = 0$, flow will take place with $\dot{u}_\alpha$ normal to the limit surface at the load point and in the outward direction if the normal vector is uniquely determined and between adjacent normals if it is not. $\psi(P_\alpha)$ will be a convex function and loads states such that $\psi(P_\alpha) > 0$ cannot be supported with finite velocities $\dot{u}_\alpha$.

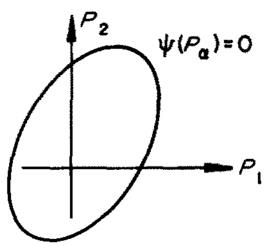

FIG. 2.

Now consider dynamic behavior at one instant t. Let the external loads be $\hat{P}_\alpha(t)$ and let the velocities be $\dot{u}_\alpha(t)$. The beam is in motion and consequently it must be *flowing* under conditions governed by the limit surface $\psi = 0$. We can easily identify *effective loads* $P_\alpha(t)$, with $\psi(P_\alpha) = 0$, such that the given velocities $\dot{u}_\alpha$ have the direction of the outward normal to the limit surface at the effective load point. It is then evident that the accelerations $\ddot{u}_\alpha$ are such that

$$\hat{P}_\alpha - m\ddot{u}_\alpha = P_\alpha. \tag{5}$$

This is shown diagrammatically in Fig. 3 for the case $\psi(\hat{P}_\alpha) > 0$; it applies equally well for $\psi(\hat{P}_\alpha) < 0$.

Two possible extremum principles are suggested by the possibility of the direct application of the limit theorems (see, for example, Prager [7]). Let P_α^* be some other load state for which $\psi(P_\alpha^*) \leq 0$, defining accelerations $\ddot{u}_\alpha^*$ such that

$$\hat{P}_\alpha - m\ddot{u}_\alpha^* = P_\alpha^*, \tag{6}$$

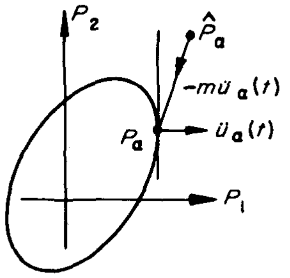

FIG. 3.

as shown in Fig. 4. The inertia forces $(-m\ddot{u}_\alpha^*)$ thus lead to any effective load which lies within or on the limit surface. Convexity of the limit surface and the normality rule require that

$$(P_\alpha - P_\alpha^*)\dot{u}_\alpha \geq 0. \tag{7}$$

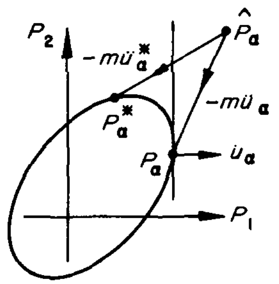

FIG. 4.

On substituting from equations (5) and (6), this gives

$$m\ddot{u}_\alpha^*\dot{u}_\alpha \geq m\ddot{u}_\alpha\dot{u}_\alpha. \tag{8}$$

If $\dot{u}_\alpha$ is given for the beam, the strain rates in the beam are known, and hence the rate of dissipation of energy in plastic work, which we shall denote by

$$\int_S D(\dot{u}_\alpha)\,\mathrm{d}s$$

can be computed. We can then define effective loads P_α^0 and a class of accelerations $\ddot{u}_\alpha^0$ by means of the equations

$$P_\alpha^0 = \hat{P}_\alpha - m\ddot{u}_\alpha^0 \tag{9}$$

and

$$(\hat{P}_\alpha - m\ddot{u}_\alpha^0)\dot{u}_\alpha = \int_S D(\dot{u}_\alpha)\,\mathrm{d}s. \tag{10}$$

These effective load states lie along the line which is tangent to limit surface at the point where $\dot{u}_\alpha$ is the normal vector (Fig. 5). There is, however, no way in which we can select the

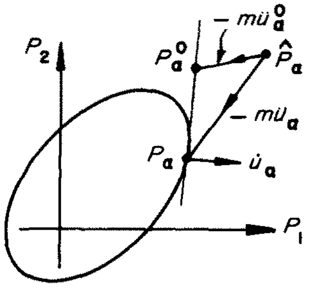

FIG. 5.

actual effective load P_α from the class P^0_α without further consideration of the problem.

The model suggests one extremum principle for the problem, a generalization of the result given in equation (8). Let us proceed now to formulate and prove this result for the general case.

4. AN EXTREMUM PRINCIPLE FOR THE ACCELERATION FIELD IN A RIGID-PLASTIC DYNAMIC PROBLEM

We return now to the general formulation of the problem in the terms given in Section 2. Suppose that at an instant t we know the loads $\hat{\mathbf{p}}(s,t)$ on S_p, the velocities $\dot{\mathbf{u}}(s,t)$ on S_p and $\dot{\mathbf{u}} = 0$ on S_u. The strain rate field $\dot{q}_j(s,t)$ can be derived from the velocity field. We wish to find the acceleration field $\ddot{\mathbf{u}}(s,t)$ at this same instant.

Let $\ddot{\mathbf{u}}^*(s,t)$ be an acceleration field such that $\{\hat{\mathbf{p}}(s,t) - m\ddot{\mathbf{u}}^*(s,t)\}$ is dynamically admissible with stresses $Q_j^*(s,t)$ which are such that $\phi(Q_j^*) \leq 0$. $\ddot{\mathbf{u}}^*(s,t)$ can thus be referred to as a *safe, dynamically admissible acceleration field*. Although it is not strictly necessary, we can limit the class of safe, dynamically admissible acceleration field by requiring that $\ddot{\mathbf{u}}^*(s,t) = 0$ on S_u.

The actual acceleration field $\ddot{\mathbf{u}}(s,t)$ must be contained within the class of dynamically admissible acceleration fields. It is distinguished from other members of the class in that the associated stresses $Q_j(s,t)$ are associated with the strain rates $\dot{q}_j(s,t)$, derived from the given velocities $\dot{\mathbf{u}}(s,t)$, through the constitutive relation (2).

Now formulate the functional

$$J(\ddot{\mathbf{u}}^*) = \int_S m\ddot{\mathbf{u}}^* \cdot \dot{\mathbf{u}}\, \mathrm{d}s. \tag{11}$$

We shall show that J takes on its least value when $\ddot{\mathbf{u}} = \ddot{\mathbf{u}}^*$. It is noted that

$$J(\ddot{\mathbf{u}}^*) - J(\ddot{\mathbf{u}}) = \int_S m\ddot{\mathbf{u}}^* \cdot \dot{\mathbf{u}}\, \mathrm{d}s - \int_S m\ddot{\mathbf{u}} \cdot \dot{\mathbf{u}}\, \mathrm{d}s. \tag{12}$$

The principle of virtual velocities gives directly that

$$\int_S \hat{\mathbf{p}} \,.\, \dot{\mathbf{u}} \, ds - \int_S m\ddot{\mathbf{u}}^* \,.\, \dot{\mathbf{u}} \, ds = \int_S Q_j^* \dot{q}_j \, ds \tag{13a}$$

$$\int_S \hat{\mathbf{p}} \,.\, \dot{\mathbf{u}} \, ds - \int_S m\ddot{\mathbf{u}} \,.\, \dot{\mathbf{u}} \, ds = \int_S Q_j \dot{q}_j \, ds. \tag{13b}$$

Subtracting equation (13a) from (13b), it is evident that

$$\int_S m\ddot{\mathbf{u}}^* \,.\, \dot{\mathbf{u}} \, ds - \int_S m\ddot{\mathbf{u}} \,.\, \dot{\mathbf{u}} \, ds = \int_S (Q_j - Q_j^*)\dot{q}_j \, ds. \tag{14}$$

Since $\phi(Q_j^*) \leq 0$ and Q_j, $\dot{q}_j$ are associated through the constitutive relation when $\dot{q}_j \neq 0$, it follows [equation (3)], that

$$(Q_j - Q_j^*)\dot{q}_j \geq 0. \tag{15}$$

Hence, substituting equations (14) and (15) into (12), it follows that

$$J(\ddot{\mathbf{u}}^*) \geq J(\ddot{\mathbf{u}}). \tag{16}$$

While our present approach does not provide a complementary or dual extremum principle, such a principle has been given by Tamuž [8] and further discussed by Reitman [9]. In this principle we adopt a velocity field $\dot{\mathbf{u}}^0(s, t^1)$, defined for $t^1 \geq t$ and satisfying the boundary conditions on S_u. We require that $\dot{\mathbf{u}}^0(s, t) = \dot{\mathbf{u}}(s, t)$, where $\dot{\mathbf{u}}(s, t)$ is the known velocity field at time t. From the function $\dot{\mathbf{u}}^0(s, t^1)$ we may derive a kinematically defined acceleration field $\ddot{\mathbf{u}}^0(s, t)$. Similarly from the time derivative of the strain rate field $\dot{q}_j^0(s, t^1)$, derived from $\dot{\mathbf{u}}^0(s, t^1)$, we may find the strain acceleration $\ddot{q}_j^0(s, t)$. This class contains the actual acceleration field $\ddot{\mathbf{u}}(s, t)$ and the actual strain acceleration $\ddot{q}_j(s, t)$. Tamuž's principle then states that the functional

$$J(\ddot{\mathbf{u}}^0) = \int_S \tfrac{1}{2} m\ddot{\mathbf{u}}^0 \,.\, \ddot{\mathbf{u}}^0 \, ds - \int \hat{\mathbf{p}} \,.\, \ddot{\mathbf{u}}^0 \, ds + \int Q_j^0 \ddot{q}_j^0 \, ds \tag{17}$$

takes its least value when $\ddot{\mathbf{u}}^0 = \ddot{\mathbf{u}}$. In this expression Q_j^0 is *the stress associated with the strain rate* $\dot{q}_j(s, t)$ *in the deforming region of the body*. If $\dot{q}_j \neq 0$ everywhere in the body, $Q_j(s)$ is fully determined and it can be seen that the Euler equation associated with the requirement that the first variation of J should be zero is simply the dynamic equation. Normally, however, regions will exist where $\dot{q}_j(s, t) = 0$, and in these rigid regions the stress is not determined by the strain rate. In the rigid region Q_j^0 is taken to be the stress associated with the strain acceleration $\ddot{q}_j^0$ through the constitutive equation when $\ddot{q}_j^0$ is treated as if it were a strain rate. This process is permissible since the stress is determined by the ratio of the components of strain rate and not its absolute magnitude. Note that Q_j will be the stress associated with $\ddot{q}_j$ when $\dot{q}_j = 0$, since plastic flow will be about to occur at that point with the strain rate having the direction of $\ddot{q}_j$.

In general the velocity fields in rigid plastic dynamic problems will involve propagating discontinuities, and great care must be taken in computing $\ddot{\mathbf{u}}^0(s, t)$ and $\dot{q}_j^0(s, t)$. For completeness we will give a brief proof of Tamuž's principle; this proof does not rigorously deal with the discontinuities in that we shall treat $\ddot{\mathbf{u}}^0(s, t)$, $\ddot{q}_j^0(s, t)$ and $\ddot{\mathbf{u}}(s, t)$, $\ddot{q}_j(s, t)$ as kinematically admissible (i.e. satisfying the strain rate–velocity relations) which in general they do not. It is adequate, however, to give some indication of the relation between Tamuž's principle and the results we shall obtain later.

First, by appeal to the principal of virtual velocities, we see that

$$J(\ddot{\mathbf{u}}) = \int_S \tfrac{1}{2}m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s - \int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s + \int_S Q_j\ddot{q}_j\,\mathrm{d}s = -\tfrac{1}{2}\int_s m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s. \tag{18}$$

Further, again from the principal of virtual velocities,

$$0 = \int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s - \int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s + \int_S Q_j\ddot{q}_j^*\,\mathrm{d}s. \tag{19}$$

Subtracting equation (18) from (17), and adding (19),

$$J(\ddot{\mathbf{u}}^0) - J(\ddot{\mathbf{u}}) = \int_S \tfrac{1}{2}m(\ddot{\mathbf{u}} - \ddot{\mathbf{u}}^0)\,.\,(\ddot{\mathbf{u}} - \ddot{\mathbf{u}}^0)\,\mathrm{d}s + \int (Q_j^0 - Q_j)\ddot{q}_j^0\,\mathrm{d}s. \tag{20}$$

The first term in this expression is non-negative. When $\dot{q}_j \neq 0$, $Q_j = Q_j^0$ and when $\dot{q}_j = 0$

$$(Q_j^0 - Q_j)\ddot{q}_j^0 \geq 0. \tag{21}$$

Consequently

$$J(\ddot{\mathbf{u}}^0) \geq J(\ddot{\mathbf{u}}). \tag{22}$$

The somewhat artificial way of defining Q_j^0 is essential if we are to get the correct answer in this extremum principle. It is in this definition that $\dot{\mathbf{u}}(s, t)$ enters the principle. If Q_j^0 were defined simply as the stress associated with $\ddot{q}_j^0$ an extremum principle could still be obtained, but it would not give the acceleration field for the desired velocity field. This point can be clarified by comparing this result with another extremum principle we shall obtain in Section 5.

The result involving safe dynamically admissible acceleration fields presented in this section is similar to a minimum principle given for rigid-viscoplastic structures by Nielsen [10]. Nielsen's result would appear to be valid for rigid, perfectly plastic materials, but involves a more restricted class of acceleration fields. Since we shall confine ourselves to results suggested by the simple model, Nielsen's result will not be developed further.

5. MODE SOLUTIONS

In terms of the model discussed in Section 3, consider a dynamic problem in which the external loads $\hat{P}_\alpha$ are *constant*, i.e. they do not vary with time. For given initial velocities $\dot{u}_\alpha$ let us consider the variation of the effective load P_α with time, confining ourselves first to load states which lie *outside* the limit surface, such that $\psi(\hat{P}_\alpha) \geq 0$.

Figure 6 shows the constant loads $\hat{P}_\alpha$, the initial velocity $\dot{u}_\alpha(0)$ and the initial value of the effective load. It is clearly seen that the acceleration $\ddot{u}_\alpha(0)$ is such as to rotate the velocity vector $\dot{u}_\alpha$, during an interval of time dt, so that the effective load P_α tends to a value P_α^m which is such that the outward normal vector to the yield surface passes through $\hat{P}_\alpha$. During the following increments of time P_α will continue to move around the limit surface until it becomes equal to P_α^m. Consequently *P_α^m is the asymptotic solution for P_α*. When $P_\alpha = P_\alpha^m$, the inertia forces $-m\ddot{u}_\alpha$ are constant and consequently the accelerations $\ddot{u}_\alpha = \ddot{u}_\alpha^m$ are

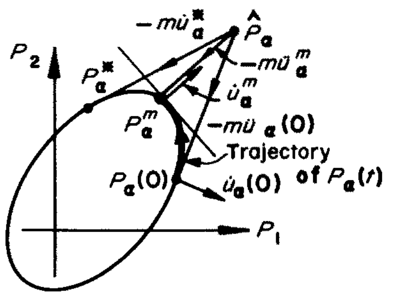

FIG. 6.

constant. Further $\dot{u}_\alpha = \dot{u}^m_\alpha$ has the same direction as $\ddot{u}_\alpha$, and *must be a linear function of time.* The asymptotic solution is thus a *mode solution* of the form

$$\dot{u}_\alpha = \ddot{u}^m_\alpha(a+t) \tag{23}$$

where a is a constant. The use of mode solutions as a basis for approximate methods was discussed by Martin and Symonds [1]; in this context it is of considerable interest to have extremum principles which will provide the acceleration field $\ddot{u}^m_\alpha$ in the asymptotic solution directly.

The forms of two complementary extremum principles are suggested by the model. First, let P^*_α be any effective load such that $\psi(P^*_\alpha) \leq 0$. It can be seen from the diagram that the magnitude of the vector $(\hat{P}_\alpha - P^*_\alpha)$ is greater than the magnitude of the vector $(\hat{P}_\alpha - P^m_\alpha)$. With a view of generalizing this result (which is in a somewhat simple form because the two masses in the model are equal), we can state this equivalently by saying that the component of $m\ddot{u}^*_\alpha$ in the direction of $\ddot{u}^*_\alpha$ is greater than the component of $m\ddot{u}^m_\alpha$ in the direction of $\ddot{u}^m_\alpha$, where $\ddot{u}^*_\alpha$ is any acceleration vector such that $\psi(\hat{P}_\alpha - m\ddot{u}^*_\alpha) \leq 0$. Thus

$$m\ddot{u}^*_\alpha\ddot{u}^*_\alpha \geq m\ddot{u}^m_\alpha\ddot{u}^m_\alpha. \tag{24}$$

Alternatively, suppose we treat any choice of acceleration vector $\ddot{u}^0_\alpha$ as a velocity vector and associate with it a rate of dissipation of energy in the beam denoted by

$$\int_S D(\ddot{u}^0_\alpha)\,\mathrm{d}s.$$

By means of the equation

$$P^0_\alpha\ddot{u}^0_\alpha = (\hat{P}_\alpha - m\ddot{u}^0_\alpha)\ddot{u}^0_\alpha = \int_S D(\ddot{u}^0_\alpha), \tag{25}$$

we then define a set of effective loads which lie on a line which is tangent to the limit surface at the point where the direction of the outward normal vector has the same direction as $\ddot{u}^0_\alpha$. Let us now confine our choices of $\ddot{u}^0_\alpha$ to those for which

$$\hat{P}_\alpha\ddot{u}^0_\alpha \geq 0.$$

This class still contains $\ddot{u}^m_\alpha$. It can then be seen geometrically in Fig. 7 that $\ddot{u}^m_\alpha$ is distinguished from $\ddot{u}^0_\alpha$ in that the shortest distance from the hyperplane defined by $\ddot{u}^0_\alpha$ to the load point

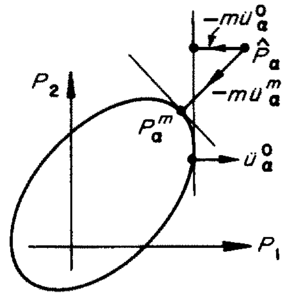

FIG. 7.

$\hat{P}_\alpha$ takes its greatest value when $\ddot{u}^0_\alpha = \ddot{u}^m_\alpha$. This can be written as

$$m\ddot{u}^0_\alpha\ddot{u}^0_\alpha \leq m\ddot{u}^m_\alpha\ddot{u}^m_\alpha. \tag{26}$$

In order to include all the elements which are used in the computation of $\ddot{u}^0_\alpha$, it may be noted from equation (25) that

$$\tfrac{1}{2}m\ddot{u}^0_\alpha\ddot{u}^0_\alpha = \hat{P}_\alpha\ddot{u}^0_\alpha - \int_S D(\ddot{u}^0_\alpha)\,\mathrm{d}s - \tfrac{1}{2}m\ddot{u}^0_\alpha\ddot{u}^0_\alpha. \tag{27}$$

Thus, substituting from equation (27), inequality (26) may be written as

$$\int_S D(\ddot{u}^0_\alpha)\,\mathrm{d}s - \hat{P}_\alpha\ddot{u}^0_\alpha + \tfrac{1}{2}m\ddot{u}^0_\alpha\ddot{u}^0_\alpha \geq \int_S D(\ddot{u}^m_\alpha)\,\mathrm{d}s - \hat{P}_\alpha\ddot{u}^m_\alpha + \tfrac{1}{2}m\ddot{u}^m_\alpha\ddot{u}^m_\alpha. \tag{28}$$

The simple model thus suggests two extremum principles, one based on the dynamic requirements and the other on the kinematic requirements. As with other dual theorems of this type, they can be combined and written as a continued inequality; from (24) and (28),

$$\tfrac{1}{2}m\ddot{u}^*_\alpha m\ddot{u}^*_\alpha \geq \tfrac{1}{2}m\ddot{u}^m_\alpha\ddot{u}^m_\alpha = P_\alpha\ddot{u}^m_\alpha - \int_S D(\ddot{u}^m_\alpha)\,\mathrm{d}s - \tfrac{1}{2}m\ddot{u}^m_\alpha\ddot{u}^m_\alpha \geq P_\alpha\ddot{u}^0_\alpha - \int_S D(\ddot{u}^0_\alpha)\,\mathrm{d}s - \tfrac{1}{2}m\ddot{u}^0_\alpha\ddot{u}^0_\alpha. \tag{29}$$

Consider now the case where the loads $\hat{P}_\alpha$ are again constant but lie within the limit surface, so that $\psi(\hat{P}_\alpha) < 0$. Although the behavior is substantially the same as the cases already discussed, there are many important differences which affect the form of the general principles which can be established.

Figure 8 shows the case in which $\hat{P}_\alpha$ lies within the limit surface, with initial velocities $\dot{u}_\alpha(0)$. The effective load $P_\alpha(0)$ is defined and the inertia force vector $(-m\ddot{u}_\alpha)$ can be drawn.

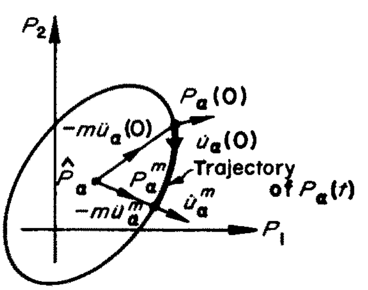

FIG. 8.

Note now that $\ddot{u}_\alpha$ is directed inwards from the limit surface while $\dot{u}_\alpha$ is directed outward. Thus the model will deccelerate, and eventually come to rest. The effective load point will move along the limit surface because $\dot{u}_\alpha$ and $\ddot{u}_\alpha$ do not have opposite directions; this process will be arrested when P_α reaches a point on the limit surface where $(-\ddot{u}_\alpha)$ has the direction of the outward normal. P_α will then remain constant during the rest of the motion, and the acceleration vector $\ddot{u}_\alpha$ will be constant, with $(-\ddot{u}_\alpha)$ proportional to $\dot{u}_\alpha$. Thus the asymptotic solution will have the form

$$\dot{u}_\alpha^m = (-\ddot{u}_\alpha^m)(b-t) \tag{30}$$

where b is a non-negative scalar. This equation holds only while $(b-t)$ is greater than zero; motion ceases when $b = t$.

While in the case where $\psi(\hat{P}_\alpha) > 0$ there was a unique effective load P_α^m describing the asymptotic solution, in the case where $\psi(\hat{P}_\alpha) < 0$ it is clear from the diagram that there exist a number of effective load P_α^m for which $(-m\ddot{u}_\alpha^m)$ has the same direction as the outward normal to the yield surface. Thus there is not a unique asymptotic solution; the asymptotic solution depends upon the initial conditions. This situation parallels that of free vibrations in an elastic structure where there are a number of modes of vibration. The asymptotic solutions are mode solutions and their number depends on $\hat{P}_\alpha$ and the shape of the limit surface.

It follows from the non-uniqueness of the mode solutions that global extremum principles cannot be expected. We can at best expect that the mode acceleration field can be identified by a local maximum or minimum of a certain functional. With this in mind, consider the dynamic theorem. Let $m\ddot{u}_\alpha^*$ be any acceleration field such that $\psi(\hat{P}_\alpha - m\ddot{u}_\alpha^*) \leq 0$. From geometric considerations, we can see that a mode solution can be distinguished from other members of this class by the characteristic that the vector $(-m\ddot{u}_\alpha)$ terminates at a point on the limit surface at which the outward normal vector has the same direction as $(-m\ddot{u}_\alpha)$. We can then argue, as shown in Fig. 9, that for small variations in the acceleration vector about a mode solution, within the class defined by the restriction $\psi(\hat{P}_\alpha - m\ddot{u}_\alpha^*) \leq 0$,

$$m\ddot{u}_\alpha \, \delta\ddot{u}_\alpha \leq 0 \quad \text{for all admissible } \delta\ddot{u}_\alpha. \tag{31}$$

This means that if we formulate the expression

$$J = \tfrac{1}{2} m\ddot{u}_\alpha^* \ddot{u}_\alpha^* \tag{32}$$

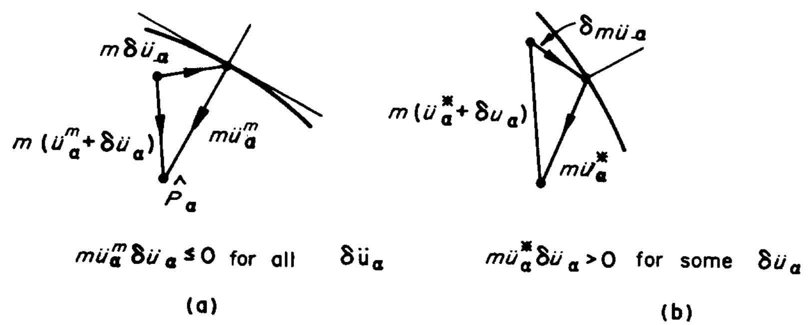

FIG. 9.

mode solutions are characterized by the condition that the first variation of J is non-positive for arbitrary variations. *This does not imply that J is a local maximum*; the inequality applies only to the first order terms in the change in J and for certain variations the first order terms will be zero. Whether J is a minimum or a maximum for this subclass of variations will then depend on the sign of the second order terms. This can be seen diagrammatically by considering a class of acceleration vectors $(-m\ddot{u}_\alpha^*)$ all of which lie on the limit surface. J, as defined in equation (31), may be either a minimum or a maximum for this class of acceleration vectors and the first variation in J is always zero. Some of the implications of this result will be discussed in greater detail in the following section, where a general proof will be given.

In formulating a complementary kinematic theorem, we note that the velocity vector has the opposite sign to the acceleration vector in the mode solution; consequently we treat $(-\ddot{u}_\alpha^0)$ as a velocity vector and define a class of acceleration vectors by the equation

$$(\hat{P}_\alpha - m\ddot{u}_\alpha^0)(-\ddot{u}_\alpha^0) = \int_S D(-\ddot{u}_\alpha^0)\,\mathrm{d}s. \tag{33}$$

The effective load point so defined will lie on a line which is normal to $\ddot{u}_\alpha^0$ and which is tangent to the yield surface or lies outside of it. If we then consider the functional

$$J = \tfrac{1}{2}m\ddot{u}_\alpha^0\ddot{u}_\alpha^0 = \int_S D(-\ddot{u}_\alpha^0)\,\mathrm{d}s + P_\alpha\ddot{u}_\alpha^0 - \tfrac{1}{2}m\ddot{u}_\alpha^0\ddot{u}_\alpha^0 \tag{34}$$

and its first variation

$$\delta J = m\ddot{u}_\alpha^0\,\delta\ddot{u}_\alpha, \tag{35}$$

we see geometrically that δJ is non-negative for an arbitrary variation only when $\ddot{u}_\alpha^0$ is a mode acceleration (Fig. 10). The first variation about a mode solution will always be zero if we restrict our variation to that subclass for which the line defined by equation (33) is tangent to the limit surface; for this class of variations J may be either a local maximum or a local minimum. The first variation will be positive definite only when the line defined by equation (33) lies outside the limit surface.

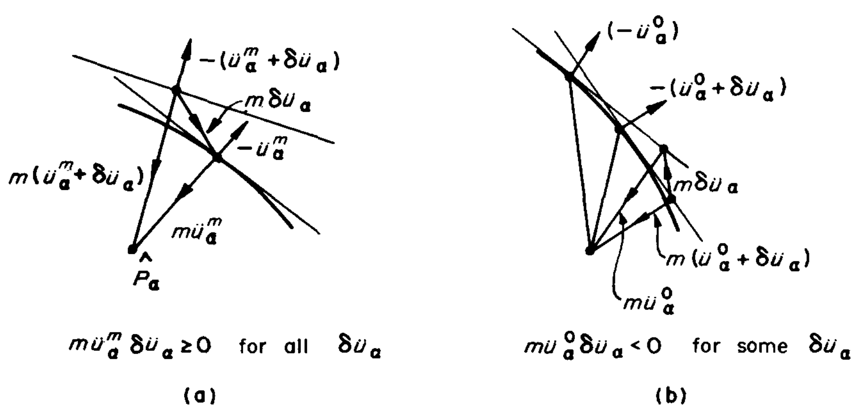

FIG. 10.

It is possible to identify the mode solution which makes J an *absolute minimum*; this mode solution can be termed the *principal mode*. In most problems the asymptotic solution will be the principal mode, and consequently it is of particular importance.

We shall now reconsider the general problem and show that the principles suggested by the simple model can be proved in the general case.

6. EXTREMUM PRINCIPLES FOR MODE SOLUTIONS

We now consider the general problem in which $\hat{\mathbf{p}}(s)$ is given on S_p and $\dot{\mathbf{u}} = 0$ on S_u. For the present the question of the asymptotic solution will be deferred; we shall simply seek *mode solutions* to the dynamic problem, i.e. solutions in which the accelerations are independent of time. Thus when $\mathbf{p}(s)$ is such that the structure cannot support the loads under quasistatic conditions, we seek solutions of the form

$$\dot{\mathbf{u}}(s, t) = \ddot{\mathbf{u}}(s)(a+t), \tag{36}$$

and when the structure can support the loads under quasistatic conditions we seek solutions of the form

$$\dot{\mathbf{u}}(s, t) = -\ddot{\mathbf{u}}(s)(b-t). \tag{37}$$

The constants a and b are both non-negative.

An alternative formulation of the mode solution, used by Martin and Symonds [1], is in many applications more convenient than equations (36) and (37), and will be presented in parallel with the approach we have hitherto taken. Here solutions of the form

$$\dot{\mathbf{u}}(s, t) = \mathbf{w}(s)T(t) \tag{38}$$

are sought, where $\mathbf{w}(s)$ is taken to have the dimensions of velocity and is referred to as the *mode shape* and $T(t)$ is dimensionless. If we denote by $\dot{q}_j^w$ the strain rates associated with the mode shape $\mathbf{w}(s)$ and note that $\dot{q}_j(s, t)$ changes only in magnitude with time, a work rate balance for the solution of equation (38) gives

$$\dot{T} = \Lambda = \frac{\int_S \hat{\mathbf{p}} \cdot \mathbf{w} \, \mathrm{d}s - \int_S D(\dot{q}_j^w) \, \mathrm{d}s}{\int_S m\mathbf{w} \cdot \mathbf{w} \, \mathrm{d}s}. \tag{39}$$

$\dot{T}$ is independent of time and hence

$$T(t) = A + \Lambda t, \qquad A > 0. \tag{40}$$

Comparing equations (38) and (40) with (36) and (37) it is evident that when $\hat{\mathbf{p}}(s)$ cannot be supported quasistatically Λ is *positive* and when $\hat{\mathbf{p}}(s)$ can be supported quasistatically Λ is negative. In both cases

$$\ddot{\mathbf{u}}(s) = \Lambda \mathbf{w}(s). \tag{41}$$

The constants which appear in the equations are related by the following expressions:

$$\begin{aligned} A &= a\Lambda \quad \text{for} \quad \Lambda > 0 \\ A &= -b\Lambda \quad \text{for} \quad \Lambda < 0. \end{aligned} \tag{42}$$

It can be noted that $\Lambda = 0$ when the loads $\hat{p}(s)$ cause quasistatic flow in the structure and that in the case $\Lambda < 0$ motion ceases when $t = t_f = -A/\Lambda$. The general results which we shall now proceed to establish will be given first in terms of the notation of equations (36) and (37) and then in the notation of equations (38)–(40).

An important feature of the mode solutions is that acceleration fields satisfy the same boundary conditions and continuity requirements as the velocity fields and that neither includes propagating discontinuities. This means that the strain accelerations $\ddot{q}_j(s)$ can be obtained simply by differentiating the acceleration field $\ddot{\mathbf{u}}(s)$ according to the strain rate displacement rate relations: it is not necessary to derive the strain rates from the velocity field and then differentiate the strain rate with respect to time. Difficulties encountered in the rigorous proof of Tamuž's principle are then no longer present. The acceleration field $\ddot{\mathbf{u}}(s)$ and the strain accelerations $\ddot{q}_j(s)$ can be rigorously treated as a kinematically admissible set in the principle of virtual velocities.

Consider first the case where $\hat{\mathbf{p}}(s)$ cannot be supported quasistatically ($\Lambda > 0$). Let us define an acceleration field $\ddot{\mathbf{u}}^*$ that satisfies the boundary conditions on S_u and is such that we can find stresses Q_j^* which are dynamically admissible with $\hat{\mathbf{p}}(s)$ on S_p, $-m\ddot{\mathbf{u}}^*$ on S_p and for which $\phi(Q_j^*) < 0$. The class of acceleration fields so defined contains the actual mode acceleration field $\ddot{\mathbf{u}}(s)$. The actual mode acceleration field is distinguished from other members of the class in that the strain accelerations $\ddot{q}_j$ derived from the acceleration field are proportional to strain rates $\dot{q}_j$ derived from the velocity field and both can be thought of as being associated with the dynamically admissible stress field Q_j (where $\dot{q}_j \neq 0$) through the constitutive relations.

Let us now set up the functional

$$J(\ddot{\mathbf{u}}^*) = \int_S \tfrac{1}{2}m\ddot{\mathbf{u}}^* . \ddot{\mathbf{u}}^* \, ds, \tag{43}$$

with

$$J(\ddot{\mathbf{u}}) = \int_S \tfrac{1}{2}m\ddot{\mathbf{u}} . \ddot{\mathbf{u}} \, ds. \tag{44}$$

By the principle of virtual velocities, treating the acceleration field as a velocity field since it satisfies the same continuity requirements,

$$\int_S (\hat{\mathbf{p}} - m\ddot{\mathbf{u}}^*)\ddot{\mathbf{u}} \, ds = \int_S Q_j^*\ddot{q}_j \, ds \tag{45a}$$

$$\int_S (\hat{\mathbf{p}} - m\ddot{\mathbf{u}})\ddot{\mathbf{u}} \, ds = \int_S Q_j\ddot{q}_j \, ds. \tag{45b}$$

Making use of the result that $(Q_j - Q_j^*)\ddot{q}_j \geq 0$, since $\phi(Q_j^*) \leq 0$ and $Q_j, \ddot{q}_j$ are associated through the constitutive relation, subtracting (45a) from (45b) gives

$$-\int_S m\ddot{\mathbf{u}} . \ddot{\mathbf{u}} \, ds \geq -\int_S m\ddot{\mathbf{u}}^* . \ddot{\mathbf{u}} \, ds. \tag{46}$$

Now, using (46),

$$J(\ddot{\mathbf{u}}^*)-J(\ddot{\mathbf{u}}) = \int_S \tfrac{1}{2}m\ddot{\mathbf{u}}^*\,.\,\ddot{\mathbf{u}}^*\,\mathrm{d}s-\int_S \tfrac{1}{2}m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s$$

$$= \int_S \tfrac{1}{2}m\ddot{\mathbf{u}}^*\,.\,\ddot{\mathbf{u}}^*\,\mathrm{d}s-\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s+\tfrac{1}{2}\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s$$

$$\geq \int_S \tfrac{1}{2}m\ddot{\mathbf{u}}^*\,.\,\ddot{\mathbf{u}}^*\,\mathrm{d}s-\int_S m\ddot{\mathbf{u}}^*\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s+\tfrac{1}{2}\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s$$

$$= \int_S \tfrac{1}{2}m(\ddot{\mathbf{u}}-\ddot{\mathbf{u}}^*)(\ddot{\mathbf{u}}-\ddot{\mathbf{u}}^*)\,\mathrm{d}s \geq 0. \tag{47}$$

This shows that $J(\ddot{u}^*)$ takes its least value when $\ddot{u}^* = \ddot{u}$. In the alternative formulation, let us choose a mode shape $w^*(s)$, satisfying the boundary conditions on S_u, and normalized by the condition

$$\int m\mathbf{w}^*\,.\,\mathbf{w}^*\,\mathrm{d}s = K \tag{48}$$

where K is a positive constant. Let Λ^* be a positive multiplier for which we can find a dynamically admissible set $-m\Lambda^*\mathbf{w}^*(s)$, $\hat{\mathbf{p}}(s)$, $Q_j^*(s)$ for which $\phi(Q_j^*) < 0$. Substituting $\Lambda^*\mathbf{w}^*$ for $\ddot{\mathbf{u}}^*$ in inequality (47), and using the fact that Λ^* and Λ are each non-negative,

$$\Lambda^* \geq \Lambda \tag{49}$$

provided that $\mathbf{w}(s)$ also satisfies the normalizing condition (48).

In order to demonstrate the kinematic theorem for the case where $\mathbf{p}(s)$ cannot be supported quasistatically ($\Lambda > 0$), we define a class of acceleration fields $\ddot{\mathbf{u}}^0(s)$ with $\ddot{\mathbf{u}}^0 = 0$ on S_u and which satisfy the same continuity requirements placed on the velocity field. This class contains the actual mode acceleration field $\ddot{\mathbf{u}}(s)$. Consider the functional

$$J(\ddot{\mathbf{u}}^0) = \int_S D(\ddot{q}_j^0)\,\mathrm{d}s-\int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s+\tfrac{1}{2}\int_S m\ddot{\mathbf{u}}^0\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s \tag{50}$$

$D(\ddot{q}_j^0) = Q_j^0\ddot{q}_j^0$ is obtained by treating $\ddot{q}_j^0$ as if it were a strain rate and determining Q_j^0 from the constitutive equation. Further

$$J(\ddot{\mathbf{u}}) = \int_S D(\ddot{q}_j)\,\mathrm{d}s-\int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s+\tfrac{1}{2}\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s. \tag{51}$$

From the principle of virtual velocities, noting that Q_j, $-m\ddot{\mathbf{u}}$ and $\hat{\mathbf{p}}$ are dynamically admissible,

$$\int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s-\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s = \int_S Q_j\ddot{q}_j\,\mathrm{d}s \tag{52a}$$

$$\int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s-\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s = \int_S Q_j\ddot{q}_j^0\,\mathrm{d}s. \tag{52b}$$

From equation (52a) we can see that

$$J(\ddot{\mathbf{u}}) = -\tfrac{1}{2}\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s. \tag{53}$$

Then, using equation (50), (52b) and (53) it follows that

$$J(\ddot{\mathbf{u}}^0)-J(\ddot{\mathbf{u}}) = \int_S \tfrac{1}{2}m(\ddot{\mathbf{u}}^0-\ddot{\mathbf{u}})\,.\,(\ddot{\mathbf{u}}^0-\ddot{\mathbf{u}})\,\mathrm{d}s+\int_S (Q_j^0-Q_j)q_j^0\,\mathrm{d}s \geq 0. \tag{54}$$

Consequently $J(\ddot{\mathbf{u}}^0)$ takes its least value when $\ddot{\mathbf{u}}^0 = \ddot{\mathbf{u}}$.

In the alternative formulation let $\mathbf{w}^0(s)$ be a mode shape for which $\mathbf{w}^0 = 0$ on S_u and which satisfies the continuity requirements for the velocity field. Let $\mathbf{w}^0$ and $\mathbf{w}$ be subjected to the normalizing condition

$$\int_S m\mathbf{w}^0\,.\,\mathbf{w}^0\,\mathrm{d}s = \int_S m\mathbf{w}\,.\,\mathbf{w}\,\mathrm{d}s = K. \tag{55}$$

Further, define Λ^0 by the condition

$$\Lambda^0 = \frac{\int_S \hat{\mathbf{p}}\,.\,\mathbf{w}^0\,\mathrm{d}s-\int_S D(\dot{q}_j^{w0})\,\mathrm{d}s}{\int m\mathbf{w}^0\,.\,\mathbf{w}^0\,\mathrm{d}s}. \tag{56}$$

We now put $\ddot{\mathbf{u}}^0 = \Lambda^0\mathbf{w}^0$ in equation (50). Using equation (56),

$$\begin{aligned} J(\Lambda^0\mathbf{w}^0) &= \Lambda^0\int_S D(q_j^{w0})\,\mathrm{d}s-\Lambda^0\int_S \hat{\mathbf{p}}\,.\,\mathbf{w}^0\,\mathrm{d}s+(\Lambda^0)^2\int_S \tfrac{1}{2}m\mathbf{w}^0\,.\,\mathbf{w}^0\,\mathrm{d}s \\ &= -(\Lambda^0)^2\int_S \tfrac{1}{2}m\mathbf{w}^0\,.\,\mathbf{w}^0\,\mathrm{d}s. \end{aligned} \tag{57}$$

Similarly,

$$J(\Lambda\mathbf{w}) = -\Lambda^2\int_S \tfrac{1}{2}m\mathbf{w}\,.\,\mathbf{w}\,\mathrm{d}s \tag{58}$$

and, using equations (54), (55) and noting that $\Lambda > 0$, we see that

$$\Lambda \geq \Lambda^0. \tag{59}$$

On considering equation (56), this result can be interpreted as stating that the mode shape maximizes the expression

$$\{\hat{\mathbf{p}}\,.\,\mathbf{w}^0\,\mathrm{d}s-\int_S D(\dot{q}_j^{w0})\,\mathrm{d}s\}$$

subject to the constraint of equation (56). Λ plays the role of a Lagrangian multiplier in this formulation.

Now consider the case where the loads $\hat{\mathbf{p}}(s)$ can be supported quasistatically ($\Lambda < 0$), and the mode solution is given by equation (37). $\ddot{\mathbf{u}}(s)$ and $\dot{\mathbf{u}}(s)$ have opposite directions, and hence we treat $(-\ddot{\mathbf{u}})$ as a velocity field and $(-\ddot{q}_j)$ as the associated strain rate field. If $Q_j(s)$ are the stresses in the mode solution, Q_j and $(-\ddot{q}_j)$ are associated through the constitutive equation. The proof of the local extremum principles closely parallels the proof of the global extremum principles already given, except that we consider only the immediate neighborhood of the mode solution.

In the dynamic theorem we define a class of acceleration fields $\ddot{\mathbf{u}}^*$, with $\ddot{\mathbf{u}}^* = 0$ on S_u and satisfying the continuity requirements on velocity fields, for which a dynamically admissible set $-m\ddot{\mathbf{u}}^*, \mathbf{p}(s), Q_j^*(s)$ with $\phi(Q_j^*) \leq 0$ can be found. This class contains *all* the mode acceleration fields. Let one mode acceleration field $\ddot{\mathbf{u}}(s)$ be selected, and formulate the functional

$$J(\ddot{\mathbf{u}}^*) = \int_S \tfrac{1}{2}m\ddot{\mathbf{u}}^* . \ddot{\mathbf{u}}^* \, \mathrm{d}s \tag{60}$$

with

$$J(\ddot{\mathbf{u}}) = \int_S \tfrac{1}{2}m\ddot{\mathbf{u}} . \ddot{\mathbf{u}} \, \mathrm{d}s. \tag{61}$$

Using the principle of virtual velocities we can write

$$\int_S (\hat{\mathbf{p}} - m\ddot{\mathbf{u}}) . (-\ddot{\mathbf{u}}) = \int_S Q_j(-\ddot{q}_j) \, \mathrm{d}s \tag{62a}$$

$$\int_S (\hat{\mathbf{p}} - m\ddot{\mathbf{u}}^*) . (-\ddot{\mathbf{u}}) \, \mathrm{d}s = \int_S Q_j^*(-\ddot{q}_j) \, \mathrm{d}s. \tag{62b}$$

From equations (60)–(62) we see that

$$J(\ddot{\mathbf{u}}) - J(\ddot{\mathbf{u}}^*) = \int_S (Q_j - Q_j^*)(-\ddot{q}_j) \, \mathrm{d}s - \tfrac{1}{2}\int_S m(\ddot{\mathbf{u}} - \ddot{\mathbf{u}}^*) . (\ddot{\mathbf{u}} - \ddot{\mathbf{u}}^*) \, \mathrm{d}s. \tag{63}$$

The integrand of each term on the right hand side of equation (63) is non-negative. In general, therefore, J does not have an extremum value when $\ddot{\mathbf{u}}^* = \ddot{\mathbf{u}}$. However, if we confine ourselves to values of the function $\ddot{\mathbf{u}}^*$ which are only infinitesimally different from $\ddot{\mathbf{u}}$, the second integrand is certainly of second order. If the first term is of first order, therefore, the second term may be neglected and the first order difference between $J(\ddot{\mathbf{u}}^*)$ and $J(\ddot{\mathbf{u}})$, which we shall term the *first variation* of $J(\ddot{\mathbf{u}})$, is non-positive. If the first term is of second order, the first variation of J is zero. We cannot assert that $J(\ddot{\mathbf{u}}^*)$ is a local minimum or maximum when $\ddot{\mathbf{u}}^* = \ddot{\mathbf{u}}$; it will often be a saddle point. However, *we can assert that a mode acceleration field is characterized by the condition that the first variation of* $J(\ddot{\mathbf{u}}^*)$ *is non-positive for arbitrary variations in the acceleration field* $\ddot{\mathbf{u}}$ *within the class* $\ddot{\mathbf{u}}^*$.

In the alternative formulation we define mode shapes $\mathbf{w}^*(s)$ which satisfy the boundary conditions on S_u, and let Λ^* be a *non-positive* multiplier for which we can find a dynamically admissible set $-m\Lambda^*\mathbf{w}^*(s), \hat{\mathbf{p}}(s), Q_j^*(s)$ with $\phi(Q_j^*) \leq 0$. Adopting the normalizing condition of equation (48), we substitute $\Lambda^*\mathbf{w}^*$ for $\ddot{\mathbf{u}}^*$ and $\Lambda\mathbf{w}$ for $\ddot{\mathbf{u}}$. Note then that

$$J(\ddot{\mathbf{u}}) - J(\ddot{\mathbf{u}}^*) = (\Lambda^2 - \Lambda^{*2})K. \tag{64}$$

Noting that both Λ and Λ^* are *non-positive* the principle enunciated above then states that, if $\mathbf{w}^*(s)$ is only infinitesimally different from $\mathbf{w}(s)$, *to first order*

$$\Lambda^* \geq \Lambda. \tag{65}$$

Note again that to first order Λ^* may be equal to Λ, while to second order Λ^* may be either greater than or less than Λ.

The form of this result is rather interesting, but it does not appear likely that it can be easily exploited as a means of finding $\mathbf{w}(s)$, at least by conventional programming methods.

In order to establish the kinematic theorem we define a class of acceleration fields $\ddot{\mathbf{u}}^0(s)$, with $\ddot{\mathbf{u}}^0 = 0$ on S_u, and which satisfy the same continuity requirements placed on the velocity field. This class contains all mode accelerations. Choose one *mode* acceleration field $\ddot{\mathbf{u}}(s)$ and formulate the functionals

$$J(\ddot{\mathbf{u}}^0) = \int_S D(-\ddot{q}_j^0)\,\mathrm{d}s + \int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s - \tfrac{1}{2}\int_S m\ddot{\mathbf{u}}^0\,.\,\ddot{\mathbf{u}}^0\,\mathrm{d}s \tag{66}$$

and

$$J(\ddot{\mathbf{u}}) = \int_S D(-\ddot{q}_j)\,\mathrm{d}s + \int_S \hat{\mathbf{p}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s - \tfrac{1}{2}\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s. \tag{67}$$

Note that $D(-q_j^0) = Q_j^0(-q_j^0)$, where Q_j^0 is the stress associated with the strain rate $(-\ddot{q}_j^0)$ and $D(-\ddot{q}_j) = Q_j(-\ddot{q}_j)$. From the principle of virtual velocities,

$$\int_S \hat{\mathbf{p}}\,.\,(-\ddot{\mathbf{u}})\,\mathrm{d}s - \int_S m\ddot{\mathbf{u}}\,.\,(-\ddot{\mathbf{u}})\,\mathrm{d}s = \int_S Q_j(-\ddot{q}_j)\,\mathrm{d}s \tag{68a}$$

$$\int_S \hat{\mathbf{p}}\,.\,(-\ddot{\mathbf{u}}^0)\,\mathrm{d}s - \int_S m\ddot{\mathbf{u}}\,.\,(-\ddot{\mathbf{u}}^0)\,\mathrm{d}s = \int_S Q_j(-\ddot{q}_j^0)\,\mathrm{d}s. \tag{68b}$$

From equation (68a) it can be seen that

$$J(\ddot{\mathbf{u}}) = +\tfrac{1}{2}\int_S m\ddot{\mathbf{u}}\,.\,\ddot{\mathbf{u}}\,\mathrm{d}s. \tag{69}$$

Then, using equations (66), (68b) and (69), it follows that

$$J(\ddot{\mathbf{u}}^0) - J(\ddot{\mathbf{u}}) = \int_S (Q_j^0 - Q_j)(-q_j^0)\,\mathrm{d}s - \int_S \tfrac{1}{2}m(\ddot{\mathbf{u}} - \ddot{\mathbf{u}}^0)\,.\,(\ddot{\mathbf{u}} - \ddot{\mathbf{u}}^0)\,\mathrm{d}s. \tag{70}$$

The integrand of each term on the right hand side of equation (70) is non-negative and consequently $J(\ddot{\mathbf{u}})$ is not a global extremum. However, if we restrict ourselves to acceleration fields $\ddot{\mathbf{u}}^0(s)$ which differ from $\ddot{\mathbf{u}}(s)$ by an infinitesimal amount, the second term is certainly of second order. If the first term is of first order, therefore, the second term may be neglected and the first order difference between $J(\ddot{\mathbf{u}}^0)$ and $J(\ddot{\mathbf{u}})$, which we shall again term the first variation of $J(\ddot{\mathbf{u}})$, is non-negative. If the first term is of second order, the first variation of J is zero. We cannot assert that J is a local minimum or maximum when $\ddot{\mathbf{u}}^0 = \ddot{\mathbf{u}}$; it will again often be a saddle point. However, we can assert that *a mode acceleration field is characterized by the condition that the first variation of* $J(\ddot{\mathbf{u}}^0)$ *is non-negative for arbitrary variations in the acceleration field* $\ddot{\mathbf{u}}$ *within the class* $\ddot{\mathbf{u}}^0$.

It may be noted that the acceleration $\ddot{\mathbf{u}}^1$ which is contained in the class $\ddot{\mathbf{u}}^0$ and which makes $J(\ddot{\mathbf{u}}^0)$ a *global* minimum satisfies the condition imposed on a mode acceleration field; let us term this the *principal mode.* It is clear then that the principal mode can be obtained by conventional programming methods. It does seem possible that other mode acceleration fields can be generated by use of the principle given above, in contrast to the dynamic principle. We offer this argument without rigorous proof; further work will be needed to establish whether it holds in sufficient generality to be useful. In many problems, particularly in impulsive loading problems were $\hat{\mathbf{p}} = 0$ on S_p, the modes are orthogonal. If this is true for the problem under consideration, it would appear that the second mode acceleration field $\ddot{\mathbf{u}}^2$ can be obtained by minimizing $J(\ddot{\mathbf{u}}^0)$ subject to the *additional* condition

$$\int m\ddot{\mathbf{u}}^0 \cdot \ddot{\mathbf{u}}^1 \, ds = 0. \tag{71}$$

The third and higher modes can then be found in a similar way. In many problems, however, the modes are not orthogonal and this procedure could not be applied.

In the alternative representation let $\mathbf{w}^0(s)$ be a mode shape, with $\mathbf{w}^0 = 0$ on S_u. Let $\mathbf{w}^0$ and w_0 again be normalized by the condition of equation (55). Further, define Λ^0 by the equation

$$\Lambda^0 = \frac{\int_S \hat{\mathbf{p}} \cdot \mathbf{w}^0 \, ds - \int_S D(\dot{q}_j^{w0}) \, ds}{\int_S m\mathbf{w}^0 \cdot \mathbf{w}^0 \, ds}. \tag{72}$$

We now replace $m\ddot{\mathbf{u}}^0(s)$ by $m\Lambda^0\mathbf{w}(s)$ in the final expression in equation (66), and $(-\ddot{\mathbf{u}}^0)$ by $\mathbf{w}^0(s)$ in the remaining places where $\ddot{\mathbf{u}}^0$ occurs. This device is necessary because Λ^0 will be negative (at least in the vicinity of Λ). Using (72), equation (66) becomes

$$J(\mathbf{w}^0) = \int_S Q_j^0 q_j^{w0} \, ds - \int_S \hat{\mathbf{p}} \cdot \mathbf{w}^0 \, ds + \Lambda^0 \int_S \tfrac{1}{2} m\mathbf{w}^0 \cdot \mathbf{w}^0 \, ds$$

$$= -\Lambda^0 \int_S \tfrac{1}{2} m\mathbf{w}^0 \cdot \mathbf{w}^0 \, ds. \tag{73a}$$

Similarly,

$$J(\mathbf{w}) = -\Lambda \int_S \tfrac{1}{2} m\mathbf{w} \cdot \mathbf{w} \, ds. \tag{73b}$$

The principle then states that to *first order*

$$\Lambda \geq \Lambda^0 \tag{74}$$

for $\mathbf{w}^0$ in the vicinity of a mode shape. Note that Λ is *negative*. When $\Lambda = \Lambda^0$ to first order, nothing further can be stated.

The *global* maximum value of Λ^0, and the mode shape associated with it, may be identified as the *principal mode*. Considering equation (72), the principle mode shape is obtained by finding the *global maximum* of the expression

$$\left\{ \int_S \hat{\mathbf{p}} \cdot \mathbf{w}^0 \, ds - \int D(\dot{q}_j^{w}) \, ds \right\}$$

subject to the constraint of equation (55). This result is similar to that obtained for the case where $\Lambda > 0$; the difference lies in that the global maximum when $\Lambda < 0$ is one of a number of local maxima or saddle points.

A resemblance between the functions of equation (50) and (66) and Tamuž's principle [equation (17)] may be observed. However, the important distinction between the principles lies in the definition of Q_j^0. In Tamuž's principle Q_j^0 is defined at least in part by the given velocity field, and the principle yields the acceleration field associated with that velocity field. In the mode extremum principles Q_j^0 is determined by $\dot{q}_j^0$ alone, and the principle yields both the mode acceleration field and the mode velocity field.

7. CONCLUSIONS

The global and local principles given in this paper provide a means of computing acceleration fields for dynamically loaded rigid-plastic structures.

The principles governing mode solutions are of particular value, complementing the work of Martin and Symonds [1] in which means of using the mode solution as an approximate method of analysis were discussed. In a future paper further applications of this approximate technique will be discussed, including applications of the principles presented in this paper.

Further study of the simple two degree of freedom model will also be carried out in the expectation that it will also provide physical insight and new results for more complex problems.

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Абстракт—Определяется некоторое число принципов зкстремума для полей ускорения, связанных с задачами жесткоге-пластической динамической нагрузки. Обращается большое внимание к глобальным и локальным принципам для видов решений, в которых можно пространственные переменные отделить от времени.

Определяются принципы зкстремума, путем исследования простой модели и затем обсуждения конструкции любой конфигурации. Все зти исследования ограничены к задачам, в которых малые перемещения.